AD-A083 484 UTAH UNIV SALT LAKE CITY DEPT OF MATHEMATICS F/G 20/1
TRANSIENT ACOUSTIC WAVE PROPAGATION IN AN EPSTEIN DUCT.(U)
NOV 79 C H WILCOX N00014-76-C-0276
UNCLASSIFIED TR-36 NL

1 OF 1
AD
A083484

END
DATE
FILMED
5 80
DTIC

6

TRANSIENT ACOUSTIC WAVE PROPAGATION IN AN EPSTEIN DUCT

C. H. Wilcox

Technical Summary Report #36

November 1979

Prepared under Contract No. N00014-76-C-0276

Task No. NR-041-370

for

Office of Naval Research

This research was supported by the Office of Naval Research.

Abstract.

Transient acoustic wave propagation is analyzed for the case of an unlimited plane-stratified fluid having constant density and sound speed $c(y)$ at depth y given by the Epstein profile

$$c^{-2}(y) = K \operatorname{sech}^2 (y/H) + L \tanh (y/H) + M$$

The acoustic potential is a solution of the wave equation

$$D_t^2 u - c^2(y) \left(D_1^2 u + D_2^2 u + D_y^2 u\right) = f(t,x,y)$$

where $x = (x_1, x_2)$ are horizontal coordinates and $f(t,x,y)$ characterizes the wave sources. The principal results of the analysis show that u is the sum of a free component, which behaves like a diverging spherical wave for large t, and a guided component which is approximately localized in a region $|y - y_0| \leq h$ and propagates outward in horizontal planes like a diverging cylindrical wave.

§1. Introduction.

This paper presents an analysis of transient acoustic wave propagation in a stationary unlimited plane stratified fluid with constant density and sound speed $c(y)$ at depth y defined by the Epstein profile

$$c(y) = [K \operatorname{sech}^2 (y/H) + L \tanh (y/H) + M]^{-1/2} \qquad (1.1)$$

where H, K, L and M are constants such that $c(y) > 0$. The acoustic field is characterized by a real-valued potential $u(t,x,y)$ that satisfies the wave equation [8]

$$D_t^2 u - c^2(y)(D_1^2 u + D_2^2 u + D_y^2 u) = f(t,x,y) \qquad (1.2)$$

where t is a time coordinate, $x = (x_1, x_2)$ are Cartesian coordinates in a horizontal plane, $f(t,x,y)$ is a function that characterizes the wave sources and $D_j = \partial/\partial x_j$, $D_t = \partial/\partial t$, $D_y = \partial/\partial y$.

The sound speed profiles (1.1) were introduced by P. S. Epstein [2] who discovered that (1.2) with this choice of $c(y)$ can be integrated by means of hypergeometric functions. This fact is of interest in theoretical acoustics because the Epstein profile provides an example of the physically interesting phenomenon of an acoustic duct when $c(y)$ has a minimum. This case is characterized by the parameter values [3,4]

$$H > 0,\ M > L \geq 0,\ K > L/2 \qquad (1.3)$$

For these values the limits $c(\pm\infty) = (M \pm L)^{-1/2}$ are finite and $c(y)$ has a unique minimum $c(y_0) < c(\infty) \leq c(-\infty)$ at $y_0 = H \tanh^{-1}(L/2K)$.

The integration of equation (1.2) below is based on the spectral theory of the Epstein operator A in the Hilbert space $\mathcal{H} = L_2(R^3, c^{-2}(y)dxdy)$ as developed in [3,4] and the notation and results of that paper are used. Equation (1.2) is interpreted as the equation

$$D_t^2 u + Au = f(t,\cdot), \quad t \in R, \tag{1.4}$$

for a function $t \to u(t,\cdot) \in \mathcal{H}$. The wave sources are assumed to act during the time interval $[0,T]$, so that supp $f \subset [0,T]$. The corresponding acoustic wave is the solution of (1.4) that satisfies the initial condition

$$u(t,\cdot) = 0 \text{ for all } t < 0 \tag{1.5}$$

The solution is given by Duhamel's integral

$$u(t,\cdot) = \int_0^t \{A^{-1/2} \sin (t - \tau) A^{1/2}\} f(\tau,\cdot)d\tau, \quad t \geq 0 \tag{1.6}$$

Indeed, if $f \in C([0,T],\mathcal{H})$ then (1.6) is the unique "solution with finite energy" of [9], while if $f \in C([0,T], D(A^{1/2}))$ then (1.6) is the "strict solution with finite energy." In addition, if $f \in C([0,T], D(A^{-1/2}))$ then

$$u(t,x,y) = \text{Re}\,\{v(t,x,y)\} \tag{1.7}$$

where $v(t,\cdot)$ is the complex-valued potential defined by

$$v(t,\cdot) = i \exp\{-it A^{1/2}\} A^{-1/2} \int_0^t \exp\{i\tau A^{1/2}\} f(\tau,\cdot)d\tau$$

In particular,

$$v(t,\cdot) = \exp\{-it\, A^{1/2}\}h, \quad t \geq T \tag{1.8}$$

where

$$h = i\, A^{-1/2} \int_0^T \exp\{i\tau\, A^{1/2}\}\, f(\tau,\cdot)d\tau \tag{1.9}$$

The initial-value problem

$$D_t^2 u + Au = 0 \text{ for all } t > 0 \tag{1.10}$$

$$u(0) = f, \; D_t u(0) = g \tag{1.11}$$

can be treated by the same formalism. Indeed, if $f \in D(A^{1/2})$ and $g \in D(A^{-1/2})$ then the solution of (1.10), (1.11) is given by (1.7), (1.8) where $h = f + i\, A^{-1/2} g \in D(A^{1/2})$ (cf. [11, Ch.3]).

The integral

$$E(u,K,t) = \int_K \{(D_1 u)^2 + (D_2 u)^2 + (D_y u)^2 + c^{-2}(y)(D_t u)^2\}dxdy \tag{1.12}$$

may be interpreted as the energy of the acoustic field u in the set $K \subset R^3$ at time t. Moreover, A is the selfadjoint operator associated with the Dirichlet integral

$$A(u,v) = \sum_{j=1}^{2} (D_j u, D_j v)_{L_2(R^3)} + (D_y u, D_y v)_{L_2(R^3)}$$

in the sense of T. Kato's theory of sesquilinear forms [5]. Indeed, if the domain of A is $D(A) = L_2^1(R^3)$, the first Sobolev space, then A is precisely the associated operator of Kato's theory. It follows from Kato's second representation theorem [5, p. 331] that

$$D(A^{1/2}) = L_2^1(R^3) \tag{1.13}$$

and

$$\|A^{1/2} u\|_{\mathcal{H}}^2 = A(u,u) = \sum_{j=1}^{2} \|D_j u\|_{L_2(R^3)}^2 + \|D_y u\|_{L_2(R^3)}^2 \tag{1.14}$$

Hence the total energy satisfies

$$E(u,R^3,t) = \|A^{1/2} u\|_{\mathcal{H}}^2 + \|D_t u\|_{\mathcal{H}}^2 \tag{1.15}$$

If $h \in D(A^{1/2})$ and u is defined by (1.7), (1.8) then a simple calculation shows that

$$E(u,R^3,t) = \|A^{1/2} h\|_{\mathcal{H}}^2 \text{ for all } t \geq T \tag{1.16}$$

The analysis of the structure of the acoustic potential (1.7), (1.8) presented below is based on the eigenfunction expansion of [4]. The orthogonal projections in $\mathcal{H}$

$$P_f = P_+ + P_- + P_0 \tag{1.17}$$

and

$$P_g = \sum_{k=1}^{\infty} P_k \tag{1.18}$$

defined by the eigenfunction expansion provide a decomposition

$$u(t,\cdot) = u_f(t,\cdot) + u_g(t,\cdot)$$

into orthogonal partial waves

$$u_f(t,\cdot) = P_f\, u(t,\cdot), \qquad u_g(t,\cdot) = P_g\, u(t,\cdot)$$

The first, called the free component, will be shown to behave for large times like a diverging spherical wave in a homogeneous fluid. The

second, called the guided component, will be shown to be approximately localized near the plane $y = y_0$ and to propagate outward in horizontal planes like a diverging cylindrical wave. This second component shows the profound effect of an acoustic duct on transient acoustic waves. It is absent when $c(y)$ has no minimum.

Transient wave propagation in the analogous but simpler case of the Pekeris duct ($c(y)$ piece-wise constant) was analyzed by Wilcox in [12] and [13]. This paper is a sequel to the report [10] of Wilcox and the article [4] of Guillot and Wilcox. In [10] the special case of a symmetric Epstein profile ($L = 0$) was analyzed without detailed proofs. [4] presented a complete spectral analysis of the Epstein operator. Here the results of [4] are used to treat the general Epstein duct and to supply the proofs that were omitted in [10]. Some of the results of this paper were announced in [1].

The remainder of this paper is organized as follows. In §2 the eigenfunction expansion of [4] is reformulated to provide a convenient starting point for the analysis of the free component u_f. The behavior of $u_f(t,\cdot)$ for $t \to \infty$ is calculated in §3. The justification of these calculations is technically more difficult than for the Pekeris profile treated in [13] and proofs of the results are presented here for the first time. §4 presents a calculation of the asymptotic behavior for $t \to \infty$ of the guided component $u_g(t,\cdot)$. §5 presents applications of the results of §§3 and 4 to calculating asymptotic distributions of energy for large times. The proofs of the results in §§4 and 5 are essentially the same as those for the Pekeris profile, given in [13], and are therefore omitted. A version of the method of stationary phase for oscillatory integrals containing parameters in given in an Appendix.

§2. Eigenfunction Expansions for the Epstein Operator.

The generalized eigenfunctions $\psi_\pm(x,y,p,\lambda)$ and $\psi_0(x,y,p,\lambda)$ were defined in [4] as multiples of certain hypergeometric functions. The normalizing constants $a_\pm(p,\lambda)$ and $a_0(p,\lambda)$ are determined only up to factors of modulus 1. In [4] these factors were chosen to make $a_\pm(p,\lambda)$ and $a_0(p,\lambda)$ real and positive [4, pp. 92-93]. Here it will be convenient to renormalize to make the constants $c_\pm(p,\lambda)$ and $c_0(p,\lambda)$ of the asymptotic forms [4, (1.7), (1.8), (1.11)] real and positive. Calculation of $c_\pm(p,\lambda)$ and $c_0(p,\lambda)$ when $a_\pm(p,\lambda)$ and $a_0(p,\lambda)$ are defined as in [4] shows that the quantities $(4\pi q_\pm)^{1/2}\, c_\pm(p,\lambda) = \exp\{i\, \Psi_\pm(p,\lambda)\}$ and $(4\pi q_+)^{1/2}\, c_0(p,\lambda) = \exp\{i\, \Psi_0(p,\lambda)\}$ have modulus 1. Hence, if $\psi_\pm$ and ψ_0 are renormalized by replacing $a_\pm(p,\lambda)$ and $a_0(p,\bar{\lambda})$ by $a_\pm(p,\lambda)\exp\{-i\, \Psi_\pm(p,\lambda)\}$ and $a_0(p,\lambda)\exp\{-i\, \Psi_0(p,\lambda)\}$ then

$$(4\pi q_\pm)^{1/2}\, c_\pm(p,\lambda) = 1, \qquad (4\pi q_+)^{1/2}\, c_0(p,\lambda) = 1 \tag{2.1}$$

Explicit expressions for $\exp\{i\, \Psi_\pm(p,\lambda)\}$ and $\exp\{i\, \Psi_0(p,\lambda)\}$ as quotients of products of Γ-functions are easily obtained but will not be needed here. The normalization (2.1) is employed in the remainder of this paper.

The generalized eigenfunctions $\psi_\pm$ and ψ_0, renormalized as above, satisfy

$$\psi_\pm(y,p,\lambda) = (4\pi q_\pm)^{-1/2}\, T_\pm(y,p,\lambda)\exp\{\mp i\, q_\mp y\} \tag{2.2}$$

$$\psi_0(y,p,\lambda) = (4\pi q_+)^{-1/2}\, T_0(y,p,\lambda)\exp\{\tilde{\beta}\, y/H\} \tag{2.3}$$

where

$$T_+(y,p,\lambda) = (4\pi q_+)^{1/2}\, a_+(p,\lambda)\, \exp\{-i\Psi_+(p,\lambda) + iq_- y\}\, \phi_3(y,p,\lambda)$$

$$T_-(y,p,\lambda) = (4\pi q_-)^{1/2}\, a_-(p,\lambda)\, \exp\{-i\Psi_-(p,\lambda) - iq_+ y\}\, \phi_4(y,p,\lambda)$$

$$T_0(y,p,\lambda) = (4\pi q_+)^{1/2}\, a_0(p,\lambda)\, \exp\{-i\Psi_0(p,\lambda) - \tilde{\beta} y/H\}\, \phi_1(y,p,\lambda)$$

and

$$\lim_{y\to\mp\infty} T_\pm(y,p,\lambda) = T_\pm(p,\lambda) \tag{2.4}$$

$$\lim_{y\to-\infty} T_0(y,p,\lambda) = T_0(p,\lambda) \tag{2.5}$$

Similarly, well-known identities for the hypergeometric functions imply [3]

$$\psi_\pm(y,p,\lambda) = (4\pi q_\pm)^{-1/2}\{I_\pm(y,p,\lambda)\exp(\mp iq_\pm y) + R_\pm(y,p,\lambda)\exp(\pm iq_\pm y)\} \tag{2.6}$$

$$\psi_0(y,p,\lambda) = (4\pi q_+)^{-1/2}\{I_0(y,p,\lambda)\exp(-iq_+ y) + R_0(y,p,\lambda)\exp(iq_+ y)\} \tag{2.7}$$

where

$$I_+(y,p,\lambda) = (2/H)^{1/2}\exp(iq_+ y)\ \phi_2(y,p,\lambda)$$

$$I_-(y,p,\lambda) = (2/H)^{1/2}\exp(-iq_- y)\ \phi_1(y,p,\lambda)$$

$$I_0(y,p,\lambda) = (2/H)^{1/2}\exp(iq_+ y)\ \phi_2(y,p,\lambda)$$

$$R_+(y,p,\lambda) = (2/H)^{1/2}\exp(-iq_+ y)\ R_+(p,\lambda)\ \phi_4(y,p,\lambda)$$

$$R_-(y,p,\lambda) = (2/H)^{1/2}\exp(iq_- y)\ R_-(p,\lambda)\ \phi_3(y,p,\lambda)$$

$$R_0(y,p,\lambda) = (2/H)^{1/2}\exp(-iq_+ y)\ R_0(p,\lambda)\ \phi_4(y,p,\lambda)$$

and

$$\lim_{y\to\pm\infty} I_{\pm}(y,p,\lambda) = 1 \tag{2.8}$$

$$\lim_{y\to+\infty} I_0(y,p,\lambda) = 1 \tag{2.9}$$

$$\lim_{y\to\pm\infty} R_{\pm}(y,p,\lambda) = R_{\pm}(p,\lambda) \tag{2.10}$$

$$\lim_{y\to+\infty} R_0(y,p,\lambda) = R_0(p,\lambda) \tag{2.11}$$

The easily calculated expressions for $R_{\pm}(p,\lambda)$, $R_0(p,\lambda)$, $T_{\pm}(p,\lambda)$ and $T_0(p,\lambda)$ as quotients of products of Γ-functions were given in [3] and will not be needed here. They imply that

$$q_{\pm}\ |R_{\pm}|^2 + q_{\mp}\ |T_{\pm}|^2 = q_{\pm}, \qquad |R_0| = 1 \tag{2.12}$$

The construction of $u_f(t,\cdot) = P_f\ u(t,\cdot)$ will be simplified by representing $P_f = P_+ + P_- + P_0$ by means of a single family of generalized eigenfunctions, rather than the three families ψ_+, ψ_- and ψ_0. This may be motivated by noting that ψ_+, ψ_- and ψ_0, collectively, represent the response of the Epstein fluid to the incident plane waves $\exp\{i(p\cdot x - q\ y)\}$, $(p,q) \in R^3$. To see this consider the mappings

$$(p,q) = X_+(p,\lambda) = (p,q_+(p,\lambda)), \qquad (p,\lambda) \in \Omega$$

$$(p,q) = X_0(p,\lambda) = (p,q_+(p,\lambda)), \qquad (p,\lambda) \in \Omega_0$$

$$(p,q) = X_-(p,\lambda) = (p,-q_-(p,\lambda)), \qquad (p,\lambda) \in \Omega$$

X_+ is an analytic transformation of Ω onto the cone

$$C_+ = \{(p,q):\ \ q > a|p|\}$$

where

$$a = ((c(-\infty)/c(\infty))^2 - 1)^{1/2} \geq 0$$

Similarly, X_0 is an analytic transformation of Ω_0 onto the cone

$$C_0 = \{(p,q):\ \ 0 < q < a|p|\}$$

and X_- is an analytic transformation of Ω onto the cone

$$C_- = \{(p,q):\ \ q < 0\}$$

Thus, the asymptotic forms of $\psi_\pm$ and ψ_0 for $y \to \pm\infty$ [4,(1.7), (1.8), (1.11)] show that $\psi_+(x,y,p,\lambda)$ with $(p,\lambda) \in \Omega$ is the response of the medium to a plane wave $\exp\{i(p \cdot x - q\ y)\}$ with $(p,q) \in C_+$, $\psi_0(x,y,p,\lambda)$ is the response to a plane wave with $(p,q) \in C_0$ and $\psi_-(x,y,p,\lambda)$ is the response to a plane wave with $(p,q) \in C_-$. Note that

$$R^3 = C_+ \cup C_0 \cup C_- \cup N$$

where N is a Lebesgue null set.

The interpretation of ψ_+, ψ_- and ψ_0 given above suggests defining the composite eigenfunction

$$\phi_+(x,y,p,q) = (2\pi)^{-1} \exp\,(ip \cdot x)\ \ \phi_+(y,p,q),\ \ (p,q) \in C_+ \cup C_0 \cup C_- \qquad (2.13)$$

by

$$\phi_+(y,p,q) = \begin{cases} (2q)^{1/2}\, c(\infty)\ \psi_+(y,p,\lambda),\ (p,\lambda) = X_+^{-1}(p,q),\ (p,q) \in C_+ \\ (2q)^{1/2}\, c(\infty)\ \psi_0(y,p,\lambda),\ (p,\lambda) = X_0^{-1}(p,q),\ (p,q) \in C_0 \\ (2|q|)^{1/2}\, c(-\infty)\ \psi_-(y,p,\lambda),\ (p,\lambda) = X_-^{-1}(p,q),\ (p,q) \in C_- \end{cases} \qquad (2.14)$$

The terms $(2q)^{1/2}\, c(\infty)$ and $(2|q|)^{1/2}\, c(-\infty)$ are normalizing factors. Note that $2q\ c^2(\infty)$ is the Jacobian of X_+^{-1} and X_0^{-1} and $2q\ c^2(-\infty)$ is the Jacobian of X_-^{-1}.

The eigenfunction expansion of [4] will be reformulated in terms of ϕ_+. To this end let $h \in \mathcal{H}$ and note that $h_f = P_f h = P_+ h + P_- h + P_0 h$ has the representation

$$
\begin{aligned}
h_f(x,y) = & \int_\Omega \psi_+(x,y,p,\lambda)\ \tilde{h}_+(p,\lambda)\ dpd\lambda \\
& + \int_{\Omega_0} \psi_0(x,y,p,\lambda)\ \tilde{h}_0(p,\lambda)\ dpd\lambda \\
& + \int_\Omega \psi_-(x,y,p,\lambda)\ \tilde{h}_-(p,\lambda)\ dpd\lambda
\end{aligned}
$$

where the integrals converge in $\mathcal{H}$. The $\mathcal{H}$-lim notation will be suppressed for brevity. Changing the variables in the three integrals by means of X_+, X_0 and X_-, respectively, gives

$$
\begin{aligned}
h_f(x,y) = & \int_{C_+} \phi_+(x,y,p,q)\ \tilde{h}_+(p,\lambda)\ c(\infty)\ (2q)^{1/2}\, dpdq \\
& + \int_{C_0} \phi_+(x,y,p,q)\ \tilde{h}_0(p,\lambda)\ c(\infty)\ (2q)^{1/2}\, dpdq \\
& + \int_{C_-} \phi_+(x,y,p,q)\ \tilde{h}_-(p,\lambda)\ c(-\infty)(2|q|)^{1/2}\, dpdq \\
= & \int_{R^3} \phi_+(x,y,p,q)\ \hat{h}_+(p,q)\ dpdq
\end{aligned}
$$

where

$$\lambda = \lambda(p,q) = \begin{Bmatrix} c^2(\infty)(|p|^2 + q^2), & q > 0 \\ c^2(-\infty)(|p|^2 + q^2), & q < 0 \end{Bmatrix} \tag{2.15}$$

and

$$\hat{h}_+(p,q) = \begin{Bmatrix} (2q)^{1/2}\, c(\infty)\, \tilde{h}_+(p,\lambda(p,q)), & (p,q) \in C_+ \\ (2q)^{1/2}\, c(\infty)\, \tilde{h}_0(p,\lambda(p,q)), & (p,q) \in C_0 \\ (2|q|)^{1/2}\, c(-\infty)\, \tilde{h}_-(p,\lambda(p,q)), & (p,q) \in C_- \end{Bmatrix} \tag{2.16}$$

It is easy to verify by considering the three cones C_+, C_0 and C_- separately that

$$\hat{h}_+(p,q) = \int_{R^3} \phi_+(x,y,p,q)^* \, h(x,y)\ c^{-2}(y)\ dxdy$$

where the integral converges in $L_2(R^3)$. Moreover, it can be shown by direct calculation, using the Parseval formula of [4], that

$$\| h_f \|_{\mathcal{H}} = \| \hat{h}_+ \|_{L_2(R^3)}$$

These considerations and the eigenfunction expansion theory of [4, §5] imply

<u>Theorem 2.1</u>. For all $h \in \mathcal{H}$ the strong limit

$$\hat{h}_+(p,q) = L_2(R^3)\text{-}\lim_{M\to\infty} \int_{|x|^2+y^2\le M^2} \phi_+(x,y,p,q)^* \, h(x,y)\ c^{-2}(y)\ dxdy \tag{2.17}$$

exists. Moreover, the mapping $\Omega_+ \colon \mathcal{H} \to L_2(R^3)$ defined by $\Omega_+ h = \hat{h}_+$ is a partial isometry such that

$$\Omega_+\Omega_+^* = 1 \text{ and } \Omega_+^*\Omega_+ = P_f$$

The adjoint mapping $h_f = \Omega_+^* \hat{h}_+$ is given by

$$h_f(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{|p|^2+q^2\leq M^2} \phi_+(x,y,p,q)\ \hat{h}_+(p,q)\ dpdq \qquad (2.18)$$

Finally, Ω_+ is a spectral mapping for A in the sense that for all $h \in D(A)$ one has

$$(\Omega_+ A h)\ (p,q) = \lambda(p,q)\ \Omega_+ h(p,q) \qquad (2.19)$$

where $\lambda(p,q)$ is defined by (2.15).

Note that these results are simply a reformation of the results of [4, §5] and not a new theorem.

It is important for the calculations of §3 to recognize that another family of generalized eigenfunctions of A is defined by

$$\phi_-(x,y,p,q) = \phi_+(x,y,-p,q)^*, \quad (p,q) \in R^3 - N$$

It is clear that $A\phi_- = \lambda(p,q)\phi_-$ and

$$\phi_-(x,y,p,q) = (2\pi)^{-1} \exp\ (ip \cdot x)\ \phi_-(y,p,q)$$

Moreover,

$$\phi_-(y,p,q) = \phi_+(y,p,q)^* \qquad (2.20)$$

because $\psi_\pm(y,p,\lambda)$, $\psi_0(y,p,\lambda)$ and therefore $\phi_+(y,p,q)$, depend on p through $|p|$ alone. The asymptotic behavior of ϕ_+ and ϕ_- for $y \to \infty$ may be seen from (2.2), (2.6), (2.7), (2.14) and (2.20). It is given by

$$\phi_+(y,p,q) \sim (2\pi)^{-1/2} \begin{cases} c(\infty)\{\exp(-iqy) + R_+(p,\lambda)\exp(iqy)\}, & (p,q) \in C_+ \\ c(\infty)\{\exp(-iqy) + R_0(p,\lambda)\exp(iqy)\}, & (p,q) \in C_0 \\ c(-\infty)\; T_-(p,\lambda)\; \exp(iq_+y), & (p,q) \in C_- \end{cases}$$

and

$$\phi_-(y,p,q) \sim (2\pi)^{-1/2} \begin{cases} c(\infty)\{\exp(iqy) + R_+(p,\lambda)^*\exp(-iqy)\}, & (p,q) \in C_+ \\ c(\infty)\{\exp(iqy) + R_0(p,\lambda)^*\exp(-iqy)\}, & (p,q) \in C_0 \\ c(-\infty)\; T_-(p,\lambda)^*\exp(-iq_+y), & (p,q) \in C_- \end{cases}$$

These relations clearly imply that $\phi_-(y,p,\lambda)$ is not simply a multiple of $\phi_+(y,p,\lambda)$. By contrast, the guided mode eigenfunctions have the symmetry property

$$\psi_k(y,p) = \psi_k(y,-p)^*, \quad k = 1,2,\cdots \tag{2.21}$$

because they are real-valued and depend on p only through $|p|$.

The family $\phi_-(x,y,p,q)$, $(p,q) \in R^3 - N$, is a second family of generalized eigenfunctions for A that spans the reducing subspace $\mathcal{H}_f = P_f\mathcal{H}$. In fact, the following exact counterpart of Theorem 2.1 holds.

Theorem 2.2. For all $h \in \mathcal{H}$ the strong limit

$$\hat{h}_-(p,q) = L_2(R^3)\text{-}\lim_{M\to\infty} \int_{|x|^2+y^2\le M^2} \phi_-(x,y,p,q)^*\, h(x,y)\, c^{-2}(y)\, dxdy$$

exists. Moreover, the mapping $\Omega_-: \mathcal{H} \to L_2(R^3)$ defined by $\Omega_- h = \hat{h}_-$ is a partial isometry such that

$$\Omega_-\Omega_-^* = 1 \text{ and } \Omega_-^*\Omega_- = P_f$$

The adjoint mapping $h_f = \Omega_-^*\hat{h}_-$ is given by

$$h_f(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{|p|^2+q^2\leq M^2} \phi_-(x,y,p,q)\ \hat{h}_-(p,q)\ dpdq$$

Finally, Ω_- is a spectral mapping for A in the sense that for all $h \in D(A)$ one has

$$(\Omega_- A h)(p,q) = \lambda(p,q)\ \Omega_- h(p,q) \tag{2.22}$$

Theorem 2.2 is a direct corollary of Theorem 2.1. This follows from the observations that $f(p,q) \to f(-p,q)$ defines a unitary transformation in $L_2(R^3)$ while $f \to f^*$ defines a unitary transformation in both $\mathcal{H}$ and $L_2(R^3)$.

§3. Transient Free Waves.

The eigenfunction expansion of Theorem 2.2 is used in this section to calculate the asymptotic behavior for $t \to \infty$ of the free component $u_f(t,\cdot) = P_f\, u(t,\cdot)$. The principal result is that in each of the half-spaces R^3_+ and R^3_-, where $R^3_\pm = \{(x,y):\ \pm y > 0\}$, $u_f(t,\cdot)$ is asymptotically equal to a wave function for a homogeneous fluid with propagation speed $c(\infty)$ and $c(-\infty)$, respectively. It is this behavior that motivates the term "free component" for $u_f(t,\cdot)$.

It will be assumed that the total acoustic potential u satisfies $u(t,\cdot) = \mathrm{Re}\{v(t,\cdot)\}$ where $v(t,\cdot) = \exp\{-itA^{1/2}\}\, h$ and $h \in D(A^{1/2})$ (see §1). The corresponding partial waves $u_k(t,\cdot) = P_k\, u(t,\cdot)$ with $k \geq 1$ satisfy $u_k(t,\cdot) = \mathrm{Re}\,\{\exp\,(-itA^{1/2})\, P_k h\}$. This follows from relations (2.21) which imply that $P_k(h^*) = (P_k h)^*$. It follows by addition that $u_g(t,\cdot) = \mathrm{Re}\,\{\exp\,(-itA^{1/2})P_g h\} = \mathrm{Re}\,\{v_g(t,\cdot)\}$ and hence

$$u_f(t,\cdot) = \mathrm{Re}\,\{v_f(t,\cdot)\} \tag{3.1}$$

where $v_f(t,\cdot) = \exp\,(-itA^{1/2})P_f h = \exp\,(-itA^{1/2})h_f$. The starting point for calculating the asymptotic behavior of $u_f(t,\cdot)$ will be the eigenfunction expansions of §2. Theorems 2.1 and 2.2 imply the representations

$$v_f(t,x,y) = \int_{R^3} \phi_\pm(x,y,p,q)\, \exp\,\{-i\, t\, \lambda^{1/2}\,(p,q)\}\, \hat{h}_\pm(p,q)\, dpdq \tag{3.2}$$

convergent in $\mathcal{H}$. The $\mathcal{H}$-lim notation will be suppressed.

Equation (3.2) gives two representations of v_f corresponding to the two families ϕ_+ and ϕ_-. The calculations below are based on the ϕ_--representation which has been found to yield the simplest form of the

asymptotic wave function. It will be convenient to introduce the characteristic functions χ_+, χ_0, and χ_- of the cones C_+, C_0 and C_- in (p,q)-space and to decompose $\hat{h}_-$ as

$$\hat{h}_-(p,q) = a(p,q) + b(p,q) + c(p,q) \tag{3.3}$$

where $a = \chi_+ \hat{h}_-$, $b = \chi_0 \hat{h}_-$ and $c = \chi_- \hat{h}_-$. The corresponding decomposition of v_f is

$$v_f = v_a + v_b + v_c \tag{3.4}$$

where

$$\left.\begin{aligned} v_a &= \exp(-itA^{1/2})\Omega_-^* a \\ v_b &= \exp(-itA^{1/2})\Omega_-^* b \\ v_c &= \exp(-itA^{1/2})\Omega_-^* c \end{aligned}\right\} \tag{3.5}$$

The behavior for $t \to \infty$ of these three functions will be analyzed separately.

Behavior of v_a. The partial wave v_a has the representation

$$v_a(t,x,y) = \int_{C_+} \phi_-(x,y,p,q) \exp(-it\omega_+(p,q))\, a(p,q)\, dpdq \tag{3.6}$$

where

$$\omega_\pm(p,q) = c(\pm\infty)\sqrt{|p|^2 + q^2}$$

To discover the behavior of $v_a(t,x,y)$ for $(x,y) \in R_+^3$, $t \to \infty$ the representation (2.6) for $\phi_-(y,p,q) = (2q)^{1/2} c(\infty)\, \psi_+(y,p,\lambda)^*$ on C_+ is substituted into (3.6). The result is

$$v_a(t,x,y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp\{i(x\cdot p + yq - t\omega_+(p,q))\}\, I_+(y,p,\lambda)^* a(p,q)\, dpdq$$

$$+ \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp\{i(x\cdot p - yq - t\omega_+(p,q))\} R_+(y,p,\lambda)^* a(p,q)\, dpdq$$

It is natural to expect that in R^3_+ the partial wave $v_a(t,x,y)$ will propagate as $t \to \infty$ into regions where y is large and hence $I_+(y,p,\lambda)$ and $R_+(y,p,\lambda)$ are near their limiting values. Thus one is led to conjecture that

$$v_a(t,\cdot) \sim v^0_a(t,\cdot) + v^1_a(t,\cdot) \text{ in } L_2(R^3_+),\ t \to \infty \tag{3.7}$$

where

$$v^0_a(t,x,y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp\{i(x\cdot p + yq - t\omega_+(p,q))\}\, a(p,q)\, dpdq$$

and

$$v^1_a(t,x,y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp\{i(x\cdot p - yq - t\omega_+(p,q)\}\, R_+(p,\lambda)^* a(p,q)\, dpdq$$

$$= \frac{c(\infty)}{(2\pi)^{3/2}} \int_{-C_+} \exp\{i(x\cdot p + yq - t\omega_+(p,q))\}\, R_+(p,\lambda)^* a(p,-q)\, dpdq$$

where $-C_+ = \{(p,q):\ (p,-q) \in C_+\} = \{(p,q):\ q < -a|p|\}$. This conjecture is proved below. Note that

$$\left.\begin{aligned} v^0_a(t,\cdot) &= \exp(-it\, c(\infty)\, A_0^{1/2})\, h_a \\ v^1_a(t,\cdot) &= \exp(-it\, c(\infty)\, A_0^{1/2})\, h^1_a \end{aligned}\right\} \tag{3.8}$$

where A_0 is the selfadjoint realization in $L_2(R^3)$ of $-\Delta = -(D_1^2 + D_2^2 + D_y^2)$

and h_a and h_a^1 are the functions in $L_2(R^3)$ whose Fourier transforms are

$$\left.\begin{aligned} \hat{h}_a(p,q) &= c(\infty)\ a(p,q) = c(\infty)\ \chi_+(p,q)\ \hat{h}_-(p,q) \\ \hat{h}_a^1(p,q) &= c(\infty)R_+(p,\lambda)^* a(p,-q) = c(\infty)R_+(p,\lambda)^*(1-\chi_+(p,q))\ \hat{h}_-(p,-q) \end{aligned}\right\} \quad (3.9)$$

Both are in $L_2(R^3)$ because $\hat{h}_- \in L_2(R^3)$ and $R_+(p,\lambda)$ is bounded, by (2.12). Moreover, supp $\hat{h}_a^1 \subset -C_+$ and hence the theory of asymptotic wave functions for d'Alembert's equation [11, Ch. 2] implies that $v_a^1(t,\cdot) \sim 0$ in $L_2(R_+^3)$ when $t \to \infty$. Combining this with (3.7) gives

$$v_a(t,\cdot) \sim v_a^0(t,\cdot) \text{ in } L_2(R_+^3),\ t \to \infty \quad (3.10)$$

Now consider the behavior of $v_a(t,x,y)$ for $(x,y) \in R_-^3$, $t \to \infty$. Substituting the representation (2.2) for $\phi_-(y,p,\lambda) = (2q)^{1/2}c(\infty)\psi_+(y,p,\lambda)^*$ into (3.6) gives

$$v_a(t,x,y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp\{i(x\cdot p + yq_- - t\omega_+(p,q))\}\, T_+(y,p,\lambda)^*\, a(p,q)\, dpdq$$

which suggests the conjecture that

$$v_a(t,\cdot) \sim v_a^2(t,\cdot) \text{ in } L_2(R_-^3),\ t \to \infty \quad (3.11)$$

where

$$v_a^2(t,x,y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_+} \exp\{i(x\cdot p + yq_- - t\omega_+(p,q))\ T_+(p,\lambda)^*\, a(p,q)\, dpdq$$

Now the mapping $(p,q) \to (p,q') = X'(p,q) = (p,q_-(p,\lambda(p,q)))$ has range $X'(C_+) = R_+^3$, Jacobian $\partial(p,q)/\partial(p,q') = c^2(-\infty)q'/c^2(\infty)q$ and satisfies $\omega_+(p,q) = \omega_-(p,q')$. Thus

$$v_a^2(t,x,y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{R_+^3} \exp\{i(x \cdot p + yq' - t\omega_-(p,q'))\} \times$$

$$\times\, T_+(p,\omega_-^2(p,q'))^* \; a(p,q)(c^2(-\infty)q'/c^2(\infty)q)\; dpdq'$$

which may be written

$$v_a^2(t,\cdot) = \exp\,(-it\; c(-\infty)\; A_0^{1/2}\,)\; h_a^2$$

where $h_a^2 \in L_2(R^3)$ has Fourier transform

$$\hat{h}_a^2(p,q') = c(\infty)\; T_+(p,\omega_-^2(p,q))^* \; a(p,q(p,q'))\; c^2(-\infty)q'/c^2(\infty)q$$

Since supp $\hat{h}_a^2 \subset R_+^3$ the results of [11, Ch. 2] imply $v_a^2(t,\cdot) \sim 0$ in $L_2(R_-^3)$ when $t \to \infty$. Combining this with (3.11) gives

$$v_a(t,\cdot) \sim 0 \text{ in } L_2(R_-^3),\; t \to \infty \tag{3.12}$$

Analogous conjectures concerning $v_b(t,\cdot)$ and $v_c(t,\cdot)$ will now be formulated. Only the main steps of the calculations will be given since the method is the same as for $v_a(t,\cdot)$.

Behavior of v_b. Combining (3.5) and (2.7) gives

$$v_b(t,x,y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_0} \exp\{i(x \cdot p + yq - t\omega_+(p,q))\}\, I_0(y,p,\lambda)^*\, b(p,q)\; dpdq$$

$$+ \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_0} \exp\{i(x \cdot p - yq - t\omega_+(p,q))\}\, R_0(y,p,\lambda)^*\, b(p,q)\; dpdq$$

which suggests the conjecture

$$v_b(t,\cdot) \sim v_b^0(t,\cdot) + v_b^1(t,\cdot) \text{ in } L_2(R_+^3),\; t \to \infty \tag{3.13}$$

where

$$\left.\begin{aligned} v_b^0(t,\cdot) &= \exp\,(-it\; c(\infty)\; A_0^{1/2})\; h_b \\ v_b^1(t,\cdot) &= \exp\,(-it\; c(\infty)\; A_0^{1/2})\; h_b^1 \end{aligned}\right\} \tag{3.14}$$

and h_b and h_b^1 are the functions in $L_2(R^3)$ whose Fourier transforms are

$$\left.\begin{aligned} \hat{h}_b(p,q) &= c(\infty)\; b(p,q) = c(\infty)\; \chi_0(p,q)\; \hat{h}\_(p,q) \\ \hat{h}_b^1(p,q) &= c(\infty)\; R_0(p,\lambda)^* b(p,-q) = c(\infty)\; R_0(p,\lambda)^* \;(1-\chi_0(p,q))\; \hat{h}\_(p,-q) \end{aligned}\right\} \tag{3.15}$$

Since supp $\hat{h}_b^1 \subset -C_0 \subset R_-^3$, $v_b^1(t,\cdot) \sim 0$ in $L_3(R_+^3)$ when $t \to \infty$, by [11, Ch. 2]. Combining this with (3.13) gives

$$v_b(t,\cdot) \sim v_b^0(t,\cdot) \text{ in } L_2(R_+^3),\; t \to \infty \tag{3.16}$$

Similarly (3.5) and (2.3) imply the representation

$$v_b(t,x,y)$$

$$= \frac{c(\infty)}{(2\pi)^{3/2}} \int_{C_0} \exp\,\{i(x\cdot p - t\omega_+(p,q))\}\; T_0(y,p,\lambda)^* \exp\,(\tilde{\beta}y/H)\; b(p,q)\; dpdq$$

and since

$$\lim_{y\to-\infty} T_0(y,p,\lambda)\exp\,(\tilde{\beta}y/H) = 0$$

one expects that

$$v_b(t,\cdot) \sim 0 \text{ in } L_2(R_-^3),\; t \to \infty \tag{3.17}$$

Behavior of v_c. Combining (3.5) and (2.2) gives

$v_c(t,x,y)$

$$= \frac{c(-\infty)}{(2\pi)^{3/2}} \int_{C_-} \exp\{i(x \cdot p - yq_+ - t\omega_-(p,q))\} T_-(y,p,\lambda)^* c(p,q)\, dpdq$$

which suggests that

$$v_c(t,\cdot) \sim v_c^1(t,\cdot) \text{ in } L_2(R_+^3),\ t \to \infty \tag{3.18}$$

where

$$v_c^1(t,x,y) = \frac{c(-\infty)}{(2\pi)^{3/2}} \int_{C_-} \exp\{i(x \cdot p - yq_+ - t\omega_-(p,q))\} T_-(p,\lambda)^* c(p,q)\, dpdq$$

The mapping $(p,q) \to (p,q') = X''(p,q) = (p,-q_+(p,\lambda(p,q)))$ has range $X''(C_-) = -C_+$, Jacobian $\partial(p,q)/\partial(p,q') = c^2(\infty)q'/c^2(-\infty)q$ and satisfies $\omega_-(p,q) = \omega_+(p,q')$. Thus

$$v_c^1(t,\cdot) = \exp(-it\ c(\infty)\ A_0^{1/2})\ h_c^1$$

where h_c^1 has Fourier transform

$$\hat{h}_c^1(p,q') = c(-\infty)\ T_-(p,\omega_+^2(p,q'))^* c(p,q(p,q'))\ c^2(\infty)q'/c^2(-\infty)q$$

Moreover, supp $\hat{h}_c^1 \subset -C_+ \subset R_-^3$ and hence one expects $v_c^1(t,\cdot) \sim 0$ in $L_2(R_+^3)$, $t \to \infty$. Combining this with (3.18) gives

$$v_c(t,\cdot) \sim 0 \text{ in } L_2(R_+^3),\ t \to \infty \tag{3.19}$$

Finally, combining (3.5) and (2.2) gives

$$v_c(t,x,y) = \frac{c(-\infty)}{(2\pi)^{3/2}} \int_{C_-} \exp\{i(x \cdot p + yq - t\omega_-(p,q))\} I_-(y,p,\lambda)^* c(p,q)\, dpdq$$

$$+ \frac{c(-\infty)}{(2\pi)^{3/2}} \int_{C_-} \exp\{i(x \cdot p - yq - t\omega_-(p,q))\} R_-(y,p,\lambda)^* c(p,q)\, dpdq$$

which suggests that

$$v_c(t,\cdot) \sim v_c^0(t,\cdot) + v_c^1(t,\cdot) \text{ in } L_2(R_-^3),\ t \to \infty \tag{3.20}$$

where

$$\left.\begin{aligned} v_c^0(t,\cdot) &= \exp\ (-it\ c(-\infty)\ A_0^{1/2}\ )\ h_c \\ v_c^1(t,\cdot) &= \exp\ (-it\ c(-\infty)\ A_0^{1/2}\ )\ h_c^1 \end{aligned}\right\} \tag{3.21}$$

and h_c and h_c^1 are the functions in $L_2(R^3)$ whose Fourier transforms are

$$\left.\begin{aligned} \hat{h}_c(p,q) &= c(-\infty)\ c(p,q) = c(-\infty)\ \chi_-(p,q)\ \hat{h}_-(p,q) \\ \hat{h}_c^1(p,q) &= c(-\infty)\, R_-(p,\lambda)^*\ c(p,-q) = c(-\infty)\, R_-(p,\lambda)^*\ (1-\chi_-(p,q))\hat{h}_-(p,-q) \end{aligned}\right\} \tag{3.22}$$

Since supp $\hat{h}_c^1 \subset -C_- \subset R_+^3$, $v_c^1(t,\cdot) \sim 0$ in $L_2(R_-^3)$ when $t \to \infty$, by [11, Ch. 2]. Combining this with (3.20) gives

$$v_c(t,\cdot) \sim v_c^0(t,\cdot) \text{ in } L_2(R_-^3),\ t \to \infty \tag{3.23}$$

The asymptotic behavior of $v_f(t,\cdot)$ for $t \to \infty$ is obtained from the three cases analyzed above by superposition, equation (3.4). Thus, equations (3.10), (3.12), (3.16), (3.17), (3.19) and (3.23) imply

$$v_f(t,\cdot) \sim \left\{\begin{array}{ll} v_a^0(t,\cdot) + v_b^0(t,\cdot) & \text{in } L_2(R_+^3) \\ v_c^0(t,\cdot) & \text{in } L_2(R_-^3) \end{array}\right\}\ t \to \infty$$

Combining this with the definition of v_a^0, v_b^0 and v_c^0, equations (3.8), (3.9), (3.14), (3.15), (3.21) and (3.22) suggests

<u>Theorem 3.1</u>. For all $h \in \mathcal{H}$ let $v_f^0(t,\cdot)$ be defined by

$$v_f^0(t,x,y) = \left\{\begin{array}{l} \exp\left(-it\, c(\infty)\, A_0^{1/2}\right) h^+(x,y),\ (x,y) \in R_+^3 \\ \\ \exp\left(-it\, c(-\infty)\, A_0^{1/2}\right) h^-(x,y),\ (x,y) \in R_-^3 \end{array}\right\} \tag{3.24}$$

where h^+ and h^- are the functions in $L_2(R^3)$ whose Fourier transforms are given by

$$\hat{h}^+(p,q) = \left\{\begin{array}{ll} c(\infty)\, \hat{h}_-(p,q), & (p,q) \in R_+^3 \\ \\ 0 & ,\ (p,q) \in R_-^3 \end{array}\right\} \tag{3.25}$$

and

$$\hat{h}^-(p,q) = \left\{\begin{array}{ll} 0 & ,\ (p,q) \in R_+^3 \\ \\ c(-\infty)\, \hat{h}_-(p,q), & (p,q) \in R_-^3 \end{array}\right\} \tag{3.26}$$

Then

$$\lim_{t\to\infty} \| v_f(t,\cdot) - v_f^0(t,\cdot) \|_{\mathcal{H}} = 0 \tag{3.27}$$

Proof of Theorem 3.1. The decomposition (3.3) will be used for the proof. Moreover, for brevity, only the asymptotic equality (3.7) for $v_a(t,\cdot)$ will be proved. The five remaining cases, namely (3.11), (3.13), (3.17), (3.18) and (3.20), can be proved by the method used for v_a.

As a first step, (3.7) will be proved for the special case where $a(p,q) \in C_0^\infty(C_+)$. The general case will then be proved by using the fact that $C_0^\infty(C_+)$ is dense in $L_2(C_+)$.

When $a(p,q) \in C_0^\infty(C_+)$ the integrals defining v_a, v_a^0 and v_a^1 converge point-wise, as well as in $\mathcal{H}$, and one has

$$v_a(t,x,y) - v_a^0(t,x,y) - v_a^1(t,x,y) = \frac{c(\infty)}{(2\pi)^{3/2}} \int_{R^2} \exp(ix\cdot p)\, w(t,y,p)\, dp \tag{3.28}$$

where

$$w(t,y,p) = w^0(t,y,p) + w^1(t,y,p) \tag{3.29}$$

$$w^0(t,y,p) = \int_{a|p|}^{\infty} \exp\{i(yq - t\omega_+(p,q))\}(I_+(y,p,\lambda) - 1)^* a(p,q)\, dq \tag{3.30}$$

and

$$\begin{aligned} w^1(t,y,p) &= \int_{a|p|}^{\infty} \exp\{i(-yq - t\omega_+(p,q))\}(R_+(y,p,\lambda) - R_+(p,\lambda))^* a(p,q)\, dq \\ &= \int_{-\infty}^{-a|p|} \exp\{i(yq - t\omega_+(p,q))\}(R_+(y,p,\lambda) - R_+(p,\lambda))^* a(p,-q)\, dq \end{aligned} \tag{3.31}$$

Parseval's formula in $L_2(R^2)$, applied to (3.28), gives

$$\int_{R^2} |v_a(t,x,y) - v_a^0(t,x,y) - v_a^1(t,x,y)|^2\, dx = \frac{c^2(\infty)}{2\pi} \int_{R^2} |w(t,y,p)|^2\, dp$$

Integrating this over $y \geq 0$ gives

$$\begin{aligned} \|v_a(t,\cdot) - v_a^0(t,\cdot) - v_a^1(t,\cdot)\|_{L_2(R_+^3)} &= \frac{c(\infty)}{(2\pi)^{1/2}} \|w(t,\cdot)\|_{L_2(R_+^3)} \\ &\leq \frac{c(\infty)}{(2\pi)^{1/2}} \left(\|w^0(t,\cdot)\|_{L_2(R_+^3)} + \|w^1(t,\cdot)\|_{L_2(R_+^3)} \right) \end{aligned} \tag{3.32}$$

by (3.29) and the triangle inequality.

The estimate (3.32) implies that to prove (3.7) it is sufficient to prove that $w^0(t,\cdot) \to 0$ and $w^1(t,\cdot) \to 0$ in $L_2(R_+^3)$ when $t \to \infty$. To this end the integrals in (3.30) and (3.31) will be estimated by the method

of stationary phase as formulated in the Appendix. To apply the method define

$$r = \sqrt{t^2+y^2},\ t = r \sin\alpha,\ y = r\cos\alpha \text{ where } 0 < \alpha < \pi/2$$

and

$$\theta(p,q,\alpha) = q\cos\alpha - \omega_+(p,q)\sin\alpha \tag{3.33}$$

Then the integrals in (3.30) and (3.31) have the form (A.1) with $s = q$ $(m = 1)$ and $\xi = (p,\alpha)$ $(n = 3)$. Moreover,

$$\nabla_q \theta(p,q,\alpha) = \cos\alpha - U(p,q)\sin\alpha$$

where

$$U(p,q) = c(\infty)q/\sqrt{|p|^2+q^2}$$

is the group speed associated with the dispersion relation $\omega = \omega_+(p,q)$. Clearly $\nabla_q \theta(p,q,\alpha) = 0$ if and only if

$$y/t = \operatorname{ctn}\alpha = U(p,q)$$

Note that $\operatorname{ctn}\alpha > 0$ for $0 < \alpha < \pi/2$. In the case of (3.31), $U(p,q) < 0$ on the interval of integration and there are no points of stationary phase. In the case of (3.30), $a|p| \le q < \infty$ and hence $0 < U(p,q) < c(\infty)$. Thus there is a unique point of stationary phase if $0 < \operatorname{ctn}\alpha < c(\infty)$, or $\alpha_0 < \alpha < \pi/2$ where $\operatorname{ctn}\alpha_0 = c(\infty)$, and no point of stationary phase if $0 < \alpha \le \alpha_0$. The point of stationary phase is

$$q = \tau(p,\alpha) = |p| \operatorname{ctn}\alpha/(c^2(\infty) - \operatorname{ctn}^2\alpha)^{1/2} \tag{3.34}$$

when $0 < \operatorname{ctn}\alpha < c(\infty)$. The corresponding approximation to $w^0(t,y,p)$ is,

from (A.11),

$$w^{\infty}(t,y,p)$$

$$= \chi(t,y)\ (2\pi)^{1/2}\, g(r,p,q,\alpha) \exp\{i(yq - t\omega_{+}(p,q)) - \pi/4\}/(t\, U_q\ (p,q))^{1/2} \tag{3.35}$$

where $\chi(t,y)$ is the characteristic function of the set $\{(t,y):\ 0 < y < c(\infty)t\}$,

$$g(r,p,q,\alpha) = (I_{+}(r \cos \alpha, p, \lambda(p,q)) - 1)^{*}\ a(p,q) \tag{3.36}$$

$$U_q(p,q) = c(\infty)|p|^2/(|p|^2+q^2)^{3/2}$$

and q is given by (3.34) with ctn $\alpha = y/t$. These results and Theorems A.1 and A.2 imply the

Lemma. For every $a(p,q) \in C_0^{\infty}(C_{+})$ there exists a constant $M = M(a)$ such that

$$|w^0(t,y,p) - w^{\infty}(t,y,p)| \le M/(t^2 + y^2)^{3/4} \tag{3.37}$$

for all $t > 0$, $y > 0$ and $p \in R^2$. Similarly, for each positive integer k there exists a constant $M_k = M_k(a)$ such that

$$|w^1(t,y,p)| \le M_k/(t^2 + y^2)^{k/2} \tag{3.38}$$

for all $t > 0$, $y > 0$ and $p \in R^2$.

Proof of the Lemma. The hypothesis $a \in C_0^{\infty}(C_{+})$ implies that supp $a = K_0$ is a compact subset of C_{+}. Let K and K_1 denote its orthogonal projections on the q-axis and the p-plane, respectively. Then the functions $w^0(t,y,p)$, $w^1(t,y,p)$ and $w^{\infty}(t,y,p)$ are all zero for $p \in R^2 - K_1$. Hence it is enough to verify (3.37) and (3.38) for $p \in K_1$.

Consider first the integral in (3.31). It has the form (A.1) with $q \in \mathcal{O} = R_-$ (the negative real numbers), $(p,\alpha) \in \mathcal{O}'$ $= (R^2 - \{0\}) \times (0,\pi/2)$ and g given by

$$g(r,p,q,\alpha) = [R_+(r\cos\alpha,p,\lambda(p,q)) - R_+(p,\lambda(p,q))]^* a(p,-q) \qquad (3.39)$$

Hypothesis (A.2) is evident from (3.33), while (A.3) follows from (3.39) and well-known properties of hypergeometric functions. Moreover, $\operatorname{supp} g(r,\cdot,\cdot,\alpha) \subset K$ for all $r > 0$ and $0 < \alpha < \pi/2$ and hence (A.4) holds. The uniform estimates (A.6) may be derived from the integral representations of the hypergeometric functions - see [3, pp. 77-79]. Finally, $\theta(p,q,\alpha)$ has no points of stationary phase for $g \in K \subset R_-$. Thus Theorem A.1 is applicable to $w^1(t,y,p)$ and (3.38) follows.

Next consider the integral in (3.30). For $0 < \alpha = \operatorname{ctn}^{-1}(y/t) \leq \alpha_0$ there are no points of stationary phase $q \in K$. Moreover, $\dot{w}^\infty(t,y,p) = 0$ on this set. Hence Theorem A.1 is applicable and implies that (3.37) holds for $y \geq c(\infty)\, t$. On the other hand, for $\alpha_0 < \alpha < \pi/2$ or $0 < y < c(\infty)t$ there is exactly one point of stationary phase q given by (3.34) and hence (A.8) holds. Moreover,

$$\theta_q''(p,q,\alpha) = -U_q(p,q)\sin\alpha \neq 0 \text{ for } q \in K,\ (p,\alpha) \in \mathcal{O}'$$

which implies (A.9). Finally, (A.10) follows from properties of the hypergeometric functions, as in the preceding cases. Thus Theorem A.2 is applicable and implies that (3.37) holds for $0 < y < c(\infty)t$.

<u>Proof of Theorem 3.1 (concluded)</u>. The lemma implies that if $a \in C_0^\infty(C_+)$ then $w^0(t,\cdot)$ and $w^1(t,\cdot)$ tend to zero in $L_2(R_+^3)$ when $t \to \infty$ and thereby proves Theorem 3.1 in this case. Indeed, integrating (3.38) gives

$$\|w^1(t,\cdot)\|^2_{L_2(R^3_+)} = \int_0^\infty \int_{K_1} |w^1(t,y,p)|^2 \, dpdy$$
$$\le C_k^2 \, |K_1| \int_0^\infty (t^2 + y^2)^{-k} \, dy = C_k' \, t^{1-2k} \tag{3.40}$$

where $|K_1|$ is the measure of K_1 and $C_k' = C_k^2 \, |K_1| \, \int_0^\infty (1 + \xi^2)^{-k} \, d\xi$. Now consider w^0. The lemma implies that

$$w^0(t,y,p) = w^\infty(t,y,p) + q(t,y,p)$$

where

$$|q(t,y,p)| \le M/(t^2 + y^2)^{3/4}$$

Thus

$$\|w^0(t,\cdot)\|_{L_2(R^3_+)} \le \|w^\infty(t,\cdot)\|_{L_2(R^3_+)} + \|q(t,\cdot)\|_{L_2(R^3_+)} \tag{3.41}$$

Moreover,

$$\|q(t,\cdot)\|_{L_2(R^3_+)} \le C'/t^2$$

by the calculation used to prove (3.40) with $k = 3/2$. It remains to show that $\|w^\infty(t,\cdot)\|_{L_2(R^3_+)} \to 0$ when $t \to \infty$. Direct calculation using (3.35) and (3.36) gives

$$\|w^\infty(t,\cdot)\|_{L_2(R^3_+)} = \int_0^{c(\infty)t} \int_{K_1} |w^\infty(t,y,p)|^2 \, dpdy$$
$$= 2\pi \int_0^{c(\infty)t} \int_{K_1} |I_+(y,p,\lambda(p,q)) - 1|^2 \, |a(p,q)|^2 \, t^{-1} \, U_q(p,q)^{-1} \, dpdy \tag{3.42}$$

where $q = \tau(p, \mathrm{ctn}^{-1}(y/t))$ is given by (3.34). Since (3.34) defines the solution q of $y/t = U(p,q)$ it is natural to make the change of variable

$y = t\,U(p,q)$, $dy = t\,U_q(p,q)\,dq$ (p fixed) in (3.42). The result is

$$\|w^{\infty}(t,\cdot)\|_{L_2(R^3_+)} = 2\pi \int_{K_1}\int_{K} |I_+(t\,U(p,q),p,\lambda(p,q)) - 1|^2\, |g(p,q)|^2\, dqdp \tag{3.43}$$

since supp $g \subset K_1 \times K$. Finally the representation $I_+(y,p,\lambda) = (2/H)^{1/2} \exp(iq_+y)\,\phi_2(y,p,\lambda)$ implies that

$$I_+(t\,U(p,q),p,\lambda(p,q)) \to 1 \text{ when } t \to \infty$$

uniformly for $(p,q) \in K_1 \times K$. It follows by Lebesgue's dominated convergence theorem that $\|w^{\infty}(t,\cdot)\|_{L_2(R^3_+)} = o(1)$, $t \to \infty$. In fact, a careful estimate of the hypergeometric function ϕ_4 gives the stronger estimate

$$\|w^{\infty}(t,\cdot)\|_{L_2(R^3_+)} = O(e^{-\mu t}),\ t \to \infty$$

where $\mu = \mu(a)$ is a positive constant.

The arguments given above, applied to v_a, v_b and v_c, show that the conclusion (3.27) of Theorem 3.1 holds for all h such that $\hat{h}_- \in C_0^{\infty}(C_+ \cup C_0 \cup C_-)$. Moreover, this set is dense in $L_2(R^3)$ and hence $\Omega_-^{*} C_0^{\infty}(C_+ \cup C_0 \cup C_-)$ is dense in $\mathcal{H}_f = P_f\mathcal{H}$ by Theorem 2.2. These facts can be used to extend (3.27) to all $h \in \mathcal{H}$, provided the mappings $U(t)\colon\ \mathcal{H}_f \to L_2(R^3)$ and $U_0(t)\colon\ \mathcal{H}_f \to L_2(R^3)$ defined by $U(t)h_f = \exp(-it\,A^{1/2})h_f$ and $U_0(t)h_f = v_f^0(t,\cdot)$ are uniformly bounded for all $t \in R$. This density argument has been given in many places; see, for example, [11, Ch. 2] or [13, p. 260]. To prove the uniform boundedness note that the $L_2(R^3)$ and $\mathcal{H}$ norms are equivalent:

$$c(y_0)\ \|h\|_{\mathcal{H}} \le \|h\|_{L_2(R^3)} \le c(-\infty)\ \|h\|_{\mathcal{H}}$$

and hence

$$\|U(t)h_f\|_{L_2(R^3)} \le c(-\infty)\ \|U(t)h_f\|_{\mathcal{H}} = c(-\infty)\ \|h_f\|_{\mathcal{H}}$$

for all $t \in R$. Similarly, (3.24), (3.25) and (3.26) imply the estimates

$$\|U_0(t)h_f\|_{L_2(R^3)} = \|v_f^0(t,\cdot)\|_{L_2(R^3)}$$

$$\le (\|v_f^0(t,\cdot)\|^2_{L_2(R^3_+)} + \|v_f^0(t,\cdot)\|^2_{L_2(R^3_-)})^{1/2}$$

$$\le (\|h^+\|_{L_2(R^3)} + \|h^-\|^2_{L_2(R^3)})^{1/2}$$

$$= (\|\hat{h}^+\|^2_{L_2(R^3)} + \|\hat{h}^-\|^2_{L_2(R^3)})^{1/2}$$

$$= (c^2(\infty)\ \|\hat{h}_-\|^2_{L_2(R^3_+)} + c^2(-\infty)\ \|\hat{h}_-\|^2_{L_2(R^3_-)})^{1/2}$$

$$\le c(-\infty)\ \|\hat{h}_-\|_{L_2(R^3)} = c(-\infty)\ \|h_f\|_{\mathcal{H}}$$

for all $t \in R$. This completes the proof of Theorem 3.1.

Theorem 3.1 implies corresponding asymptotic estimates for the free component $u_f(t,\cdot) = P_f\ u(t,\cdot) = \mathrm{Re}\ \{v_f(t,\cdot)\}$ of the acoustic potential $u(t,\cdot)$. Indeed, if $u_f^0(t,\cdot)$ is defined by

$$u_f^0(t,x,y) = \mathrm{Re}\ \{v_f^0(t,x,y)\} \qquad (3.44)$$

then Theorem 3.1 and the elementary inequality $|\mathrm{Re}\ z| \le |z|$ imply

Corollary 3.2. For all $h \in \mathcal{H}$ one has

$$\lim_{t\to\infty} \| u_f(t,\cdot) - u_f^0(t,\cdot)\|_{\mathcal{H}} = 0 \tag{3.45}$$

If the initial state h has derivatives in $\mathcal{H}$ then $u_f(t,\cdot)$ and $u_f^0(t,\cdot)$ have the same derivatives in $\mathcal{H}$ and (3.45) can be strengthened to include these derivatives. In particular, one has

Corollary 3.3. For all $h \in L_2^1(R^3) = D(A^{1/2})$ one has

$$\lim_{t\to\infty} \| D_j\, u_f(t,\cdot) - D_j\, u_f^0(t,\cdot)\|_{\mathcal{H}} = 0,\ j = 0,1,2,3 \tag{3.46}$$

where $D_0 = \partial/\partial t$, $D_1 = \partial/\partial x_1$, $D_2 = \partial/\partial x_2$ and $D_3 = \partial/\partial y$. Note that (3.46) is equivalent to convergence in energy:

$$\lim_{t\to\infty} E(u_f - u_f^0, R^3, t) = 0$$

Corollary 3.3 can be proved by applying the method of this section to the derivatives $D_j\, v_f(t,\cdot)$ $(j = 0,1,2,3)$ which are given by integrals of the same form as (3.2). Detailed proofs for the case of the Pekeris profile were given in [13].

§4. Transient Guided Waves.

The asymptotic behavior for $t \to \infty$ of the guided component $u_g(t,\cdot) = P_g\, u(t,\cdot)$ is derived in this section. $u_g(t,\cdot)$ is a sum in $\mathcal{H}$ of mutually orthogonal partial waves $u_k(t,\cdot) = P_k\, u(t,\cdot) = \mathrm{Re}\,\{v_k(t,\cdot)\}$, $k \geq 1$. The starting point for the analysis is the integral representation

$$v_k(t,x,y) = \int_{\Omega_k} \psi_k(x,y,p) \exp\{-it\, \omega_k(|p|)\}\, \tilde{h}_k(p)\, dp \tag{4.1}$$

where

$$\tilde{h}_k(p) = \int_{R^3} \psi_k(x,y,p)^*\, h(x,y)\, c^{-2}(y)\, dxdy$$

and the integrals converge in $\mathcal{H}$ and $L_2(\Omega_k)$, respectively. The integral in (4.1) can be written

$$v_k(t,x,y) = \frac{1}{2\pi} \int_{\Omega_k} \exp\{i(x\cdot p - t\omega_k(|p|))\}\, \psi_k(y,p)\, \tilde{h}_k(p)\, dp \tag{4.2}$$

This is an oscillatory integral that can be estimated by the method of stationary phase of the Appendix when $\tilde{h}_k \in C_0^\infty(\Omega_k)$. To apply the method define

$$r = \sqrt{t^2 + |x|^2}$$

$$t = r\,\xi_0,\ x_1 = r\,\xi_1,\ x_2 = r\,\xi_2$$

$$\xi = (\xi_0,\xi_1,\xi_2) \in S^2 \subset R^3$$

where S^2 denote the unit sphere in R^3. Then (4.2) takes the form A.1

with m = 2, n = 3

$$\theta_k(p,\xi,y) = \xi_1 p_1 + \xi_2 p_2 - \xi_0 \omega_k(|p|) \tag{4.3}$$

$$g_k(r,p,\xi,y) = (2\pi)^{-1} \psi_k(y,p) \tilde{h}_k(p) \tag{4.4}$$

Note that θ_k is independent of y and g_k is independent of r and ξ. They have been written in the forms (4.3) and (4.4) to emphasize the applicability of the results of the Appendix.

The phase function (4.3) has a point of stationary phase if and only if

$$U_k(|p|)p/|p| = x/t \tag{4.5}$$

where

$$U_k(|p|) = D_{|p|}\, \omega_k(|p|)$$

is the group speed associated with the dispersion relation $\omega = \omega_k(|p|)$. The analysis will be simplified by the following

<u>Proposition</u>. If the parameters defining c(y), equation (1.1), satisfy

$$8K \geq M \tag{4.6}$$

then each of the functions $U_k(|p|)$, $k = 1,2,\cdots$, is a strictly decreasing function on $p_k \leq |p| < \infty$ such that $U_k(p_k) = c(\infty)$ and $U_k(|p|) \to c(y_0)$ when $|p| \to \infty$.

A proof of the proposition may be derived from the parametric representation of the dispersion relation [4]. The calculation is too long to give here.

For simplicity of presentation condition (4.6) is assumed in the remainder of this paper. If $8K < M$ then $U_k'(|p|)$ may have a finite number of zeros. This complicates the form of the asymptotic wave functions but is otherwise tractable. In the case of the Pekeris profile, treated in [13], the functions $U_k'(|p|)$ have exactly one zero on $|p| \geq p_k$.

By calculating the Hessian θ_k'' one can show that

$$r^2 \ |\det \theta_k''(p,\xi,y)| = t^2 \ U_k(|p|) \ |U_k'(|p|)|/|p|$$

and

$$\operatorname{sgn} \theta_k''(p,\xi,y) = 0$$

In particular, each point of stationary phase is non-degenerate, by the Proposition. Moreover, (4.5) implies that θ_k has a point p of stationary phase if and only if

$$|x|/t = U_k(|p|) \tag{4.7}$$

Now (4.7) has a unique solution $|p|$ if $|x|/t$ lies in the range of $U_k(|p|)$; that is if

$$c(y_0) < |x|/t < c(\infty) \tag{4.8}$$

The point of stationary phase is then given by

$$p = Q_k(|x|/t)x/|x| \ \text{for } x \in R^2 - \{0\}, \ t > 0 \tag{4.9}$$

where Q_k is the inverse function to U_k. It is a monotone decreasing function that maps $(c(y_0), c(\infty))$ onto (p_k, ∞). By Theorem A.2 the point

(4.9) makes a contribution

$$v_k^\infty(t,x,y,p) = \frac{|p|^{1/2} \exp \{i(|x||p| - t\omega_k(|p|))\} \psi_k(y,p) \tilde{h}_k(p)}{t \{U_k(|p|) |U_k'(|p|)|\}^{1/2}} \tag{4.10}$$

to the integral in (4.2) when (t,x) satisfies (4.8). For $|x|/t$ outside the interval (4.8) there is no point of stationary phase. Thus the stationary phase approximation to $v_k(t,x,y)$ is given by

$$v_k^\infty(t,x,y) = \chi(|x|/t)\ v_k^\infty(t,x,y,Q_k(|x|/t)x/|x|) \tag{4.11}$$

where χ is the characteristic function of $(c(y_0),c(\infty))$ and one has

Theorem 4.1. For all $h \in \mathcal{H}$ such that $\tilde{h}_k \in C_0^\infty(\Omega_k)$ there exists a constant $C = C_k(h)$ such that

$$|v_k(t,x,y) - v_k^\infty(t,x,y)| \le C/t^2 \tag{4.12}$$

for all $t > 0$, $x \in R^2 - \{0\}$ and $y \in R$.

This result can be proved by application of Theorems A.1 and A.2. The proof is similar to that of the Lemma in §3 and will not be recorded here. If $\tilde{h}_k$ is not a smooth function then Theorems A.1 and A.2 are not applicable and the estimate (4.12) may fail. However, the definition (4.10), (4.11) is applicable to all $h \in \mathcal{H}$. More precisely, one can prove

Theorem 4.2. For all $h \in \mathcal{H}$, all $t > 0$ and $k = 1,2,\cdots$ one has

$$v_k^\infty(t,\cdot) \in \mathcal{H},$$

and

$$\| v_k^\infty(t,\cdot)\|_{\mathcal{H}} = \|\tilde{h}_k\|_{L_2(\Omega_k)} = \|P_k h\|_{\mathcal{H}}$$

Moreover, the mapping $t \to v_k^\infty(t,\cdot)$ is continuous from R_+ to $\mathcal{H}$ and

$$\lim_{t\to\infty} \| v_k(t,\cdot) - v_k^\infty(t,\cdot)\|_{\mathcal{H}} = 0 \tag{4.13}$$

The proofs of these properties are the same as those for the Pekeris profile, given in [13], and are not reproduced here. On defining

$$u_k^\infty(t,x,y) = \operatorname{Re}\{v_k(t,x,y)\}$$

one has

<u>Corollary 4.3</u>. For all $h \in \mathcal{H}$ and $k = 1,2,3,\cdots$

$$\lim_{t\to\infty} \| u_k(t,\cdot) - u_k^\infty(t,\cdot)\|_{\mathcal{H}} = 0$$

If $h \in L_2^1(R^3)$ then $u_k(t,\cdot) \in L_2^1(R^3)$ and asymptotic wave functions for the first derivatives of u_k can be constructed. Indeed, if $\tilde{h}_k \in C_0^\infty(\Omega_k)$ then the first derivatives of v_k are given by

$$D_t\, v_k(t,x,y) = \frac{1}{2\pi}\int_{\Omega_k} \exp\{i(x\cdot p - t\omega_k(|p|))\}\, \psi_k(y,p)(-i\omega_k(|p|))\, \tilde{h}_k(p)\, dp$$

$$D_j\, v_k(t,x,y) = \frac{1}{2\pi}\int_{\Omega_k} \exp\{i(x\cdot p - t\omega_k(|p|))\}\, \psi_k(y,p)(ip_j)\, \tilde{h}_k(p)\, dp \quad (j = 1,2)$$

$$D_y\, v_k(t,x,y) = \frac{1}{2\pi}\int_{\Omega_k} \exp\{i(x\cdot p - t\omega_k(|p|))\}\, D_y\, \psi_k(y,p)\, \tilde{h}_k(p)\, dp$$

These integrals have the same form as the integral (4.2) for $v_k(t,x,y)$. The corresponding asymptotic wave functions are defined by

$$v_{k0}^\infty(t,x,y,p) = (-i\omega_k(|p|))\, v_k^\infty(t,x,y,p)$$

$$v_{kj}^\infty(t,x,y,p) = (ip_j)\, v_k^\infty(t,x,y,p) \qquad (j = 1,2)$$

$$v_{k3}^{\infty}(t,x,y,p) = D_y \, v_k^{\infty}(t,x,y,p)$$

and

$$v_{kj}^{\infty}(t,x,y) = \chi(|x|/t) \, v_{kj}^{\infty}(t,x,y,Q_k(|x|/t)x/|x|) \quad (j = 0,1,2,3)$$

The analogue of Theorem 4.2 is

<u>Theorem 4.4</u>. For all $h \in L_2^1(R^3)$, all $t > 0$ and $k = 1,2,3,\cdots$ one has

$$v_{k0}^{\infty}(t,\cdot) \in \mathcal{H}, \; v_{kj}^{\infty}(t,\cdot) \in L_2(R^3), \; j = 1,2,3$$

$$\| v_{k0}^{\infty}(t,\cdot) \|_{\mathcal{H}}^2 + \sum_{j=1}^{3} \| v_{kj}^{\infty}(t,\cdot) \|_{L_2(R^3)}^2 = 2 \, \| A^{1/2} h_k \|_{\mathcal{H}}^2$$

and

$$\lim_{t\to\infty} \| D_j \, v_k(t,\cdot) - v_{kj}^{\infty}(t,\cdot) \|_{\mathcal{H}} = 0 \quad (j = 0,1,2,3)$$

Finally if

$$u_{kj}^{\infty}(t,x,y) = \mathrm{Re}\, \{v_{kj}^{\infty}(t,x,y)\} \quad (j = 0,1,2,3)$$

then one has

<u>Corollary 4.5</u>. For all $h \in L_2^1(R^3)$ and $k = 1,2,3,\cdots$

$$\lim_{t\to\infty} \| D_j \, u_k(t,\cdot) - u_{kj}^{\infty}(t,\cdot) \|_{\mathcal{H}} = 0$$

Theorem 4.4 may be proved by the technique used for the Pekeris profile in [13].

§5. The Asymptotic Distribution of Energy for Large Times.

The total energy of the acoustic field $u(t,x,y)$, given by (1.16), is constant for $t \geq T$. The same is true of the partial waves u_f, u_g and u_k, $k = 1,2,\cdots$. Moreover, the eigenfunction expansion of [4] implies that $\{P_f, P_1, P_2, \cdots\}$ is a complete family of orthogonal projections that reduces A. It follows that

$$\|A^{1/2} h\|^2_{\mathcal{H}} = \|A^{1/2} h_f\|^2_{\mathcal{H}} + \sum_{k=1}^{\infty} \|A^{1/2} h_k\|^2_{\mathcal{H}} \tag{5.1}$$

which may be interpreted as an energy partition theorem. The energies of the partial waves can be calculated from the initial state h, or source function $f(t,x,y)$, by means of the eigenfunction expansion theorem. The relationship between h and f is given by (1.9). Hence

$$\hat{h}_-(p,q) = i\, \lambda^{-1/2}(p,q)\; \hat{f}_-(-\lambda^{1/2}(p,q),p,q)$$

where

$$\hat{f}_-(\omega,p,q) = \int_{-\infty}^{\infty} \int_{R^3} \exp\,(-i\omega\tau)\; \phi_-(x,y,p,q)^*\; f(\tau,x,y)\; c^{-2}(y)\; dydx$$

and similarly

$$\tilde{h}_k(p) = i\, \omega_k^{-1}(|p|)\; \tilde{f}_k(-\omega_k(|p|),p)$$

where

$$\tilde{f}_k(\omega,p) = \int_{-\infty}^{\infty} \int_{R^3} \exp\,(-i\omega\tau)\; \psi_k(x,y,p)^*\; f(\tau,x,y)\; c^{-2}(y)\; dydx$$

It follows that the partial energies are given by

$$E(u_f,R^3,t) = \|A^{1/2} h_f\|^2_{\mathcal{H}} = \int_{R^3} |\hat{f}_-(-\lambda^{1/2}(p,q),p,q)|^2 \, dpdq, \quad t \geq T$$

and

$$E(u_k,R^3,t) = \|A^{1/2} h_k\|^2_{\mathcal{H}} = \int_{\Omega_k} |\tilde{f}_k(-\omega_k(|p|),p)|^2 \, dp, \quad t \geq T$$

The theorems of §3 and §4 make possible the calculation of asymptotic energy distributions in bounded and unbounded subsets of R^3. Only the principal results are formulated here since the proofs are the same as for the case of the Pekeris profile given in [13].

The notation

$$E^\infty(u,K) = \lim_{t\to\infty} E(u,K,t)$$

will be used whenever the limit exists. A first result is the transiency of all waves with finite energy in an Epstein duct:

$$E^\infty(u,K) = 0 \text{ for all compact sets } K \subset R^3$$

Next let C^+ (resp. C^-) denote a cone in R^3_+ (resp. R^3_-). Then Corollary 3.3 and the results of [11] imply

$$E^\infty(u_f,C^\pm) = c^2(\pm\infty) \int_{C^\pm} (|p|^2 + q^2) \, |\hat{h}_-(p,q)|^2 \, dpdq$$

$$= \int_{C^\pm} |\hat{f}_-(-\lambda^{1/2}(p,q),p,q)|^2 \, dpdq$$

Now define the cone C_ε by

$$C_\varepsilon = \{(x,y):\ |y - y_0| \le \varepsilon |x|\},\ \varepsilon > 0$$

Then, in contrast to the preceding result, one has

$$\varepsilon^\infty(u_k, C_\varepsilon) = \varepsilon(u_k, R^3, t) = \| A^{1/2} h_k \|^2_{\mathcal{H}} \text{ for every } \varepsilon > 0$$

and for $k = 1,2,3,\cdots$ (see [13, Theorem 5.5]).

Finally, if

$$S = \{(x,y):\ x \in R^2 \text{ and } y_1 < y < y_2\}$$

then it can be shown by means of Corollary 4.5 that

$$\varepsilon^\infty(u_k, S) = \int_{\Omega_k} |\tilde{f}_k(-\omega_k(|p|),p)|^2 \left(\int_{y_1}^{y_2} \psi_k^2(y,p)\ c^{-2}(y)\,dy\right) dp$$

§6. Concluding Remarks. Examination of the proofs in [4] and in this paper reveals that the methods should apply to a large class of sound speed profiles c(y). As remarked in [4], the principal obstacle to such a generalization is the lack of information about the p-dependence of the eigenfunctions and eigenvalues. In the cases of the Pekeris and Epstein profiles this information was obtained from explicit constructions of the eigenfunctions in terms of known functions.

Appendix. Estimates of Oscillatory Integrals with Parameters.

The method of stationary phase provides asymptotic estimates for $r \to \infty$ of oscillatory integrals of the form

$$I(r) = \int_{\mathcal{O}} \exp\{i\, r\, \theta(s)\}\, g(s)\, ds, \quad \mathcal{O} \subset R^m$$

The method is needed in §3 and §4 above to estimate the oscillatory integrals in equations (3.30), (3.31) and (4.2). In (4.2) the integrand g contains a parameter y in addition to the variables of integration. In (3.30) and (3.31) both θ and g contain parameters and, in addition, g contains the large parameter r. In both cases estimates that are uniform in the parameters are needed. This suggests the study of oscillatory integrals of the form

$$I(r,\xi) = \int_{\mathcal{O}} \exp\{i\, r\, \theta(s,\xi)\}\, g(r,s,\xi)\, ds \tag{A.1}$$

where $r > 0$, $s \in R^m$ and $\xi \in R^n$. $\theta(s,\xi)$ is a real-valued phase function and it is assumed that there are open sets $\mathcal{O} \subset R^m$ and $\mathcal{O}' \subset R^n$ such that

$$D_s^\delta\, \theta(s,\xi) \in C(\mathcal{O} \times \mathcal{O}') \text{ for all multi-indices } \delta \tag{A.2}$$

$$D_s^\delta\, g(r,s,\xi) \in C(R_+ \times \mathcal{O} \times \mathcal{O}') \text{ for all multi-indices } \delta \tag{A.3}$$

where R_+ denotes the positive real numbers. Moreover, it is assumed that there is a compact set $K \subset \mathcal{O}$ such that

$$\operatorname{supp} g \subset R_+ \times K \times \mathcal{O}' \tag{A.4}$$

PRECEDING PAGE BLANK-NOT FILMED

Estimates of $I(r,\xi)$ for $r \to \infty$ are sought which are uniform in ξ on compact subsets of $\mathcal{O}'$. For large values of r the exponential in (A.1) is highly oscillatory except near critical points of the phase function $\theta(s,\xi)$. Two estimates of $I(r,\xi)$ are given here, corresponding to the cases of no critical points and one critical point, respectively. The first case is formulated as

Theorem A.1. Assume that

$$\nabla_s \theta(s,\xi) = (\partial\theta/\partial s_1, \cdots, \partial\theta/\partial s_m) \neq 0 \text{ for all } (s,\xi) \in K \times \mathcal{O}' \qquad \text{(A.5)}$$

Moreover, assume that for each compact set $K' \subset \mathcal{O}'$, each $r_0 > 0$ and each positive integer k there exists a constant $M = M(K,K',r_0,k) > 0$ such that

$$|D_s^\delta g(r,s,\xi)| \leq M \text{ for all } r \geq r_0,\ s \in K,\ \xi \in K' \text{ and } |\delta| \leq k \qquad \text{(A.6)}$$

Then there exists a constant $C = C(K,K',r_0,k,g) > 0$ such that

$$|I(r,\xi)| \leq C\, r^{-k} \text{ for all } r \geq r_0 \text{ and } \xi \in K' \qquad \text{(A.7)}$$

In the second case considered here $\theta(s,\xi)$ has a unique non-degenerate critical point $s = \tau(\xi)$ for each $\xi \in \mathcal{O}'$. It is formulated as

Theorem A.2. Assume that there is a function $\tau \in C^\infty(\mathcal{O}',\mathcal{O})$ such that, for all $\xi \in \mathcal{O}'$,

$$\nabla_s \theta(s,\xi) = 0 \text{ if and only if } s = \tau(\xi) \qquad \text{(A.8)}$$

Moreover, assume that the Hessian $\theta''(s,\xi) = (\partial^2\theta(s,\xi)/\partial s_j \partial s_k)$ satisfies

$$\det \theta''(\tau(\xi),\xi) \neq 0 \text{ for all } \xi \in \mathcal{O}' \qquad \text{(A.9)}$$

In addition, assume that for each compact set $K' \subset \mathcal{O}'$ and each $r_0 > 0$ there exists a constant $M = M(K,K',r_0)$ such that

$$|D_s^{\delta} g(r,s,\xi)| \leq M \text{ for all } r \geq r_0,\ s \in K,\ \xi \in K' \text{ and } |\delta| \leq m + 5 \tag{A.10}$$

Then there exists a constant $C = C(K,K',r_0,g) > 0$ such that if $q(r,\xi)$ is defined by

$$I(r,\xi) = (2\pi)^{m/2} \frac{\exp\{i\, r\, \theta(\tau(\xi),\xi) + i\frac{\pi}{4}\operatorname{sgn}\theta''(\tau(\xi),\xi)\} g(r,\tau(\xi),\xi)}{r^{m/2}\ |\det \theta''(\tau(\xi),\xi)|^{1/2}} + q(r,\xi) \tag{A.11}$$

then

$$|q(r,\xi)| \leq C\, r^{-m/2-1} \text{ for all } r \geq r_0 \text{ and } \xi \in K' \tag{A.12}$$

In (A.11), $\operatorname{sgn}\theta''(s,\xi)$ is the signature of the real symmetric matrix $\theta''(s,\xi)$.

The uniform estimates given above are due, in essence, to M. Matsumura [6]. No proofs are offered because the theorems can be proved by following Matsumura's proofs and recognizing that under the hypotheses formulated above his estimates are uniform for $\xi \in K'$.

References

1. Dermenjian, Y., J. C. Guillot and C. H. Wilcox, Comportement asymptotique des solutions de l'équation des ondes associée à l'opérateur d'Epstein, C. R. Acad. Sci. Paris 287, 731-734 (1978).

2. Epstein, P. S., Reflection of waves in an inhomogeneous absorbing medium, Proc. Nat. Acad. Sci. U. S. A. 16, 627-637 (1930).

3. Guillot, J. C., and C. H. Wilcox, Spectral analysis of the Epstein operator, Univ. of Utah Tech. Summary Rept. 27 (1975).

4. Guillot, J. C., and C. H. Wilcox, Spectral analysis of the Epstein operator, Proc. Roy. Soc. Edinburgh 80A, 85-98 (1978).

5. Kato, T., Perturbation Theory of Linear Operators, New York, Springer-Verlag 1966.

6. Matsumura, M., Asymptotic behavior at infinity for Green's functions of first order systems with characteristics of nonuniform multiplicity, Publ. RIMS, Kyoto Univ. 12, 317-377 (1976).

7. Pedersen, M. A., and D. W. White, Ray theory of the general Epstein profile, J. Acoust. Soc. Am. 44, 765-786 (1968).

8. Tolstoy, I., The theory of waves in stratified fluids including the effects of gravity and rotation, Rev. Mod. Phys. 35, 207-230 (1963).

9. Wilcox, C. H., Initial-boundary value problems for linear hyperbolic partial differential equations of the second order, Arch. Rational Mech. Anal. 10, 361-400 (1962).

10. Wilcox, C. H., Transient acoustic wave propagation in a symmetric Epstein duct, Univ. of Utah Tech. Summary Rept. 25 (1974).

11. Wilcox, C. H., Scattering Theory for the d'Alembert Equation in Exterior Domains, Lecture Notes in Mathematics, V. 442, Springer-Verlag 1975.

12. Wilcox, C. H., Spectral analysis of the Pekeris operator, Arch. Rational Mech. Anal. 60, 259-300 (1976).

13. Wilcox, C. H., Transient electromagnetic wave propagation in a dielectric waveguide, Istituto Nazionale di Alta Matematica, Symposia Mathematica XVIII, 239-277 (1976).

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

Not applicable AD-A083484

3. REPORT TITLE

(6) Transient acoustic wave propagation in an Epstein duct. 12 50

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

(9) Technical Summary Report

5. AUTHOR(S) (Last name, first name, initial)

(10) Wilcox, Calvin H. 11 Nov 79

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
November, 1979	51	13

8a. CONTRACT OR GRANT NO.

15 N00014-76-C-0276

b. PROJECT NO.

Task No. NR-041-370

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

TR #36 (14) TR-36

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

None

10. AVAILABILITY/LIMITATION NOTICES

Approved for public release, distribution unlimited

11. SUPPLEMENTARY NOTES

None

12. SPONSORING MILITARY ACTIVITY

Office of Naval Research, Code 432
Arlington, Virginia 22217

13. ABSTRACT

Abstract.

Transient acoustic wave propagation is analyzed for the case of an unlimited plane-stratified fluid having constant density and sound speed $c(y)$ at depth y given by the Epstein profile

$$c^{-2}(y) = K \operatorname{sech}^2 (y/H) + L \tanh (y/H) + M$$

The acoustic potential is a solution of the wave equation

$$D_t^2 u - c^2(y) (D_1^2 u + D_2^2 u + D_y^2 u) = f(t,x,y)$$

where $x = (x_1, x_2)$ are horizontal coordinates and $f(t,x,y)$ characterizes the wave sources. The principal results of the analysis show that u is the sum of a free component, which behaves like a diverging spherical wave for large t, and a guided component which is approximately localized in a region $|y - y_0| \leq h$ and propagates outward in horizontal planes like a diverging cylindrical wave.

DD FORM 1 JAN 64 1473

411 345

Unclassified

Security Classification

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Acoustic Wave Propagation						
Acoustic Duct						
Epstein Profile						
Normal Mode Expansion						
Guided Waves						
Free Waves						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

Unclassified

Security Classification